LAST TIME

Dot product

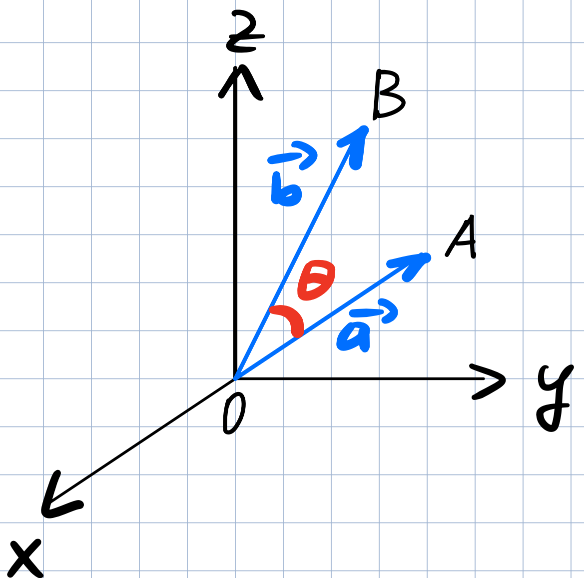
For $\vec{a} = \langle a_1, a_2, a_3 \rangle$ $\vec{b} = \langle b_1, b_2, b_3 \rangle$ two vectors,

$$\boxed{\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3}$$ the dot product

(or scalar product, or inner pr.)

Rmk: $\vec{a} \cdot \vec{a} = |\vec{a}|^2$

Q: Does $(\vec{a} \cdot \vec{b}) \cdot \vec{c}$ make sense? NO



$0 \le \theta \le \pi$ angle between vectors $\vec{a}, \vec{b}$

$(= \text{angle between } \vec{OA}, \vec{OB})$

Thm: $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$

lengths (magnitudes) angle

Corollary:

For $\vec{a}, \vec{b}$ nonzero vectors

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}$$

non-zero

$$\boxed{\vec{a} \perp \vec{b} \iff \vec{a} \cdot \vec{b} = 0}$$

Cross product

For $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$ two vectors, the vector

Works only in $\mathbb{R}^3$

$$\boxed{\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle}$$

is the cross product (or vector product)

- $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ determinant of 2x2 matrix

$$\vec{a} \times \vec{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Ex: $\vec{a} = \langle 1, 2, 3 \rangle$

$\vec{b} = \langle 2, 0, -5 \rangle$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 2 & 0 & -5 \end{vmatrix} = ((-5)2 - (0)3)\vec{i} + (3(2) - (-5)1)\vec{j} + (1(0) - (2)2)\vec{k}$$

$$= -10\vec{i} + 11\vec{j} - 4\vec{k} = \langle -10, 11, -4 \rangle$$

$$\left|\begin{array}{ccc|ccc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 & 1 & 2 & 3 \\ 2 & 0 & -5 & 2 & 0 & -5 \end{array}\right|$$

$\vec{a} = \langle 1, 2, 3 \rangle$

$\vec{b} = \langle 2, 0, -5 \rangle$

$\vec{a} \times \vec{b} = \langle -10, 11, -4 \rangle$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 2 & 0 & -5 \end{vmatrix} = (-5)2\,\vec{i} + 2(3)\,\vec{j} + (0)1\,\vec{k} - (0)3\,\vec{i} - (-5)1\,\vec{j} - (2)2\,\vec{k}$$

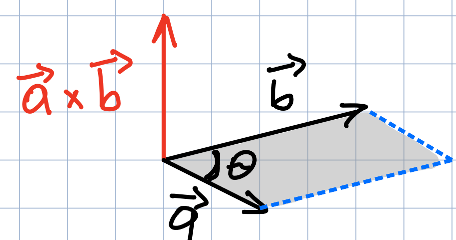
Note $(\vec{a} \times \vec{b}) \cdot \vec{a} = (-10)(1) + (11)(2) + (-4)(3) = 0$

$(\vec{a} \times \vec{b}) \cdot \vec{b} = (-10)(2) + (11)(0) + (-4)(-5) = 0$

It holds generally:

Thm: $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$

(direction of $\vec{a} \times \vec{b}$ is given by right hand rule)



Thm: $|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta$

↑ angle between $\vec{a}$ and $\vec{b}$

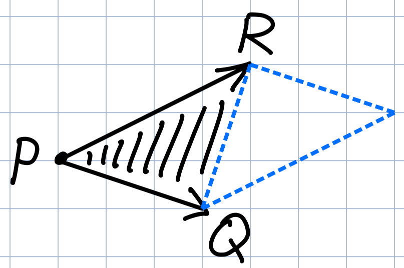
↑ area of the parallelogram

- $\vec{a}$ and $\vec{b}$ are parallel iff $\vec{a} \times \vec{b} = 0$

$$\vec{a} \times \vec{a} = 0$$

Ex: $P(-1, 2, 5)$ $\quad Q(0, 4, 8)$ $\quad R(1, 2, 0)$

Find the area of the triangle PQR



$$\overrightarrow{PQ} = \langle 1, 2, 3 \rangle$$

$$\overrightarrow{PR} = \langle 2, 0, -5 \rangle$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -10, 11, -4 \rangle$$

$$\text{Area}(PQR) = \frac{1}{2} |\overrightarrow{PQ} \times \overrightarrow{PR}| = \frac{1}{2}\sqrt{(-10)^2 + 11^2 + (-4)^2} = \frac{1}{2}\sqrt{237}$$

Properties of cross product:

1) $\vec{i} \times \vec{j} = \vec{k}$, $\quad \vec{j} \times \vec{k} = \vec{i}$, $\quad \vec{k} \times \vec{i} = \vec{j}$

$\vec{j} \times \vec{i} = -\vec{k}$, $\quad \vec{k} \times \vec{j} = -\vec{i}$, $\quad \vec{i} \times \vec{k} = -\vec{j}$

2) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

3) $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$

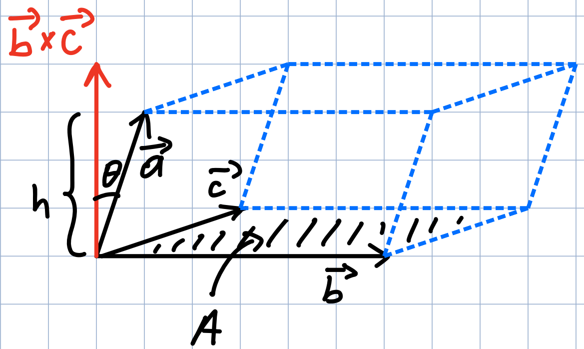
Rmk:

$(a \times b) \times c \neq a \times (b \times c)$

$(i \times i) \times j = 0 \qquad i \times \underbrace{(i \times j)}_{k} = -j$

Triple product:

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$ – (scalar) triple product of $\vec{a}, \vec{b}, \vec{c}$



Volume of the parallelepiped

$$V = A\cdot h = |\vec{b}\times\vec{c}|\,(|\vec{a}|\cos\theta)$$
$$= |\vec{a}\cdot(\vec{b}\times\vec{c})|$$

Rmk: If $V=0$ then $\vec{a}, \vec{b}, \vec{c}$ lie in the same plane (are co-planar)

Ex: $\vec{a} = \langle 1, 2, 3\rangle$ $\quad \vec{b} = \langle 4, 5, 6\rangle$ $\quad \vec{c} = \langle 7, 8, 9\rangle$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1\cdot 5\cdot 9 + 2\cdot 6\cdot 7 + 3\cdot 4\cdot 8$$
$$- 3\cdot 5\cdot 7 - 6\cdot 8\cdot 1 - 4\cdot 2\cdot 9$$
$$= 45 + 84 + 96 - 105 - 48 - 72 = 0$$

$\Rightarrow \vec{a}, \vec{b}, \vec{c}$ are co-planar